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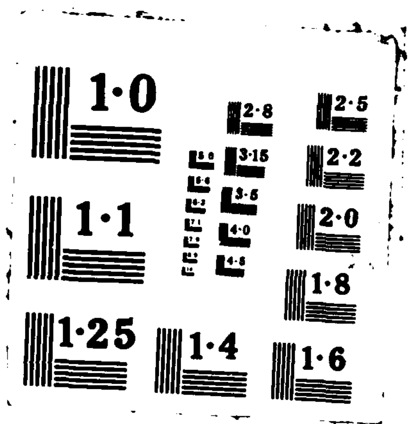
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IN CLOSE-ELLIPTICAL ORBITS

THESIS

James M Meintel
Second Lieutenant, USAF

AFIT/GA/AA/87D-3

DEPARTMENT OF THE AIR FORCE

AIR UNIVERSITY

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RELATIVE MOTION OF TWO SATELLITES IN CLOSE-ELLIPTICAL ORBITS

THESIS

Presented to the Faculty of the School of Engineering

of the Air Force Institute of Technology

Air University

In Partial Fulfillment of the

Requirements for the Degree of

Master of Science in Astronautical Engineering

James M. Meintel, B.S.E.

Second Lieutenant, USAF

November 1987

Approved for public release; distribution unlimited

Preface

The purpose of this study was to find the solution to the relative equations of motion for two satellites in close-elliptical orbits. This was done by performing a harmonic analysis on the Floquet solution, yielding an expression for the solution in terms of time and eccentricity.

The solution was first verified for the circular orbit case to confirm the accuracy of the computer code. It was then found for the elliptical orbit case for small eccentricities.

In finding the solution and writing my thesis, I received invaluable support from my faculty advisor, Dr. William Wiesel. Without his guidance this thesis would have never been completed. Finally, I would like to thank my wife Laura and daughter Kathryn for their understanding during all the extra hours that I spent with this thesis at their expense.

James M. Meintel

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Abstract

The need for a closed form solution for two bodies in close elliptical orbits is identified. Equations of motion are derived using Hamilton's equations. Floquet theory, and its applicability to this problem, is discussed. The solution for circular orbits is derived in closed form and numerically, using Floquet theory and harmonic analysis. The solution for elliptical orbits is found numerically first for very small eccentricities, finding the dependence on eccentricity. The solution is then found for all time as a function of eccentricity.

RELATIVE MOTION OF TWO SATELLITES IN CLOSE-ELLIPTICAL ORBITS

Introduction

The relative motion of satellites is a problem that has been widely studied. The results are very important when one wants to rendezvous or dock two vehicles. Historically, the first use for a solution was during the Gemini program when the first docking maneuvers were perfected for use on the Apollo missions. A complete solution to the problem is relatively simple for circular orbits; however, elliptical orbits add a degree of difficulty.

Numerous authors have published the solution to the problem of two vehicles in circular orbits, including Buning (1984) and Kaplan (1976). Buning derives the equations of motion for the elliptical case, but he limits solution to the circular case. Investigations of the elliptical problem usually amount to subtracting the position and velocity vectors of the individual bodies. This method is limited, however, because of a loss of significant digits during subtraction. Lancaster (1970) formed a computational method to calculate the relative position and velocity for elliptical orbits; however, the results are for single points in time and are not a solution to the equations of motion. It also does not include out of plane motion. Berreen and Crisp (1976) form a solution for a probe ejected into an elliptical orbit from a space station in a circular orbit, but does not address the problem of both bodies being in elliptical orbits.

The dynamics for this problem can be solved using the Lagrangian and the Hamiltonian as outlined in Meirovitch (1970). The resulting equations of motion are linear and periodic, permitting a Floquet analysis to be done. The system eigenvectors and the particular solution to the equations of motion can be expressed as functions of their Fourier coefficients, as outlined by Brouwer and Clemence (1961). The results then lead to a complete solution to the relative equations of motion expressed as an expansion of the eccentricity.

II. Problem Description

Introduction

This chapter defines the problem, equations of motion, and describes the theory used to analyze the equations of motion. The coordinate system is defined as a rotating rectangular system with its origin centered on body A. The equations of motion are derived using Lagrange's and Hamilton's equations. The resulting system is linear and time-periodic, making Floquet theory applicable in solving for the homogenous solution to the system. A brief explanation of the solution for autonomous systems is also included. In addition, this chapter contains information relevant to finding repeated eigenvectors, Fourier coefficients, and a particular solution to the equations of motion.

Equations of Motion

The equations of motion are derived by finding the Lagrangian and then forming the Hamiltonian for the system shown in Figure 1. The figure shows two coordinate systems. The first system has its origin at the center of the gravitational field, and the polar coordinates R and θ describe the orbit of body A. The vector $\underset{\sim}{\rho}$ is the position of body B with respect to A.

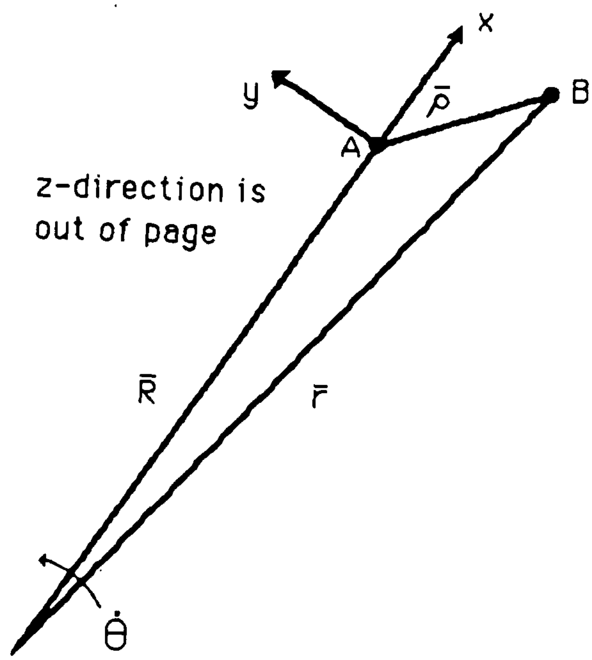

Figure 1. Physical System and Coordinate System

Both bodies are assumed to be in two-body, unperturbed orbits. The masses of each body are also assumed to be negligible, thus any gravitational attraction between bodies A and B can be ignored.

The second coordinate system is a rotating, rectangular coordinate system. The origin of the system is at the center of body A and follows body A along its two-body orbital path. The x-direction is along the position vector, $\underset{\sim}{R}$, and the y-direction is perpendicular to the x-direction in the orbital plane. The z-direction completes the right-handed system. For this system, as seen from an observer in body A, the coordinate directions are up, forward and left. The z-direction is the only coordinate out of the plane of the orbit.

In order to determine the Lagrangian for the second particle, the position and the square of the velocity vector for body B must be found. The position of body B is:

$$\underset{\sim}{r} = (R + x)\ \underset{\sim}{i} + y\ \underset{\sim}{j} + z\ \underset{\sim}{k} \tag{2.1}$$

The velocity then, including the rotating terms, is:

$$\dot{\underset{\sim}{r}} = (\dot{R} + \dot{x} - \dot{\theta}\ y)\ \underset{\sim}{i} + (\dot{y} + \dot{\theta}\ R + \dot{\theta}\ x)\ \underset{\sim}{j} + \dot{z}\ \underset{\sim}{k} \tag{2.2}$$

thus,

$$\dot{\underset{\sim}{r}} \cdot \dot{\underset{\sim}{r}} = \dot{R}^2 + \dot{x}^2 + y^2\dot{\theta}^2 + 2\ \dot{R}\ \dot{x} - 2\ \dot{\theta}\ \dot{R}\ y - 2\ \dot{\theta}\ \dot{x}\ y + \dot{y}^2$$
$$+ R^2\dot{\theta}^2 + \dot{\theta}^2 x^2 + 2\ \dot{\theta}\ x\ \dot{y} + 2\ R\ \dot{\theta}\ \dot{y} + 2\ R\ \dot{\theta}^2 x + \dot{z}^2 \tag{2.3}$$

The gravity potential per unit mass of body B for the two-body problem is:

$$V/m = -\ \mu/|\underset{\sim}{r}| = -\ \mu\ [(R + x)^2 + y^2 + z^2]^{-1/2} \tag{2.4}$$
$$= -\ \frac{\mu}{R}\left[1 + \frac{2x}{R} + \frac{x^2}{R^2} + \frac{y^2}{R^2} + \frac{z^2}{R^2}\right]^{-1/2}$$

Expanding the denominator and dropping the terms of order greater than two yields

$$V/m = -\ \frac{\mu}{R}\left[1 - \frac{x}{R} + \frac{x^2}{R^2} - \frac{y^2}{2R^2} - \frac{z^2}{2R^2} + \cdots\right] \tag{2.5}$$

With the Lagrangian per unit mass

$$\mathscr{L} = T/m - V/m \tag{2.6}$$

and the kinetic energy per unit mass

$$T/m = \frac{1}{2}\,(\dot{\underset{\sim}{r}} \cdot \dot{\underset{\sim}{r}}) \tag{2.7}$$

the resulting Lagrangian is:

$$\begin{aligned}\mathscr{L} = {} & \frac{1}{2}\dot{R}^2 + \frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{\theta}^2 y^2 + \dot{R}\,\dot{x} - \dot{\theta}\,\dot{R}\,y - \dot{\theta}\,\dot{x}\,y + \frac{1}{2}\dot{y}^2 \\ & + \frac{1}{2}\dot{\theta}^2 R^2 + \frac{1}{2}\dot{\theta}^2 x^2 + \dot{\theta}\,x\,\dot{y} + \dot{\theta}\,R\,\dot{y} + \dot{\theta}^2 R\,x + \frac{1}{2}\dot{z}^2 \\ & + \frac{\mu}{R} - \frac{\mu}{R^2}x + \frac{\mu}{R^3}x^2 - \frac{\mu}{2R^3}y^2 - \frac{\mu}{2R^3}z^2\end{aligned} \tag{2.8}$$

The Hamiltonian for a system is found from the following equation:

$$H = \sum p_i\,\dot{q}_i - \mathscr{L} \tag{2.9}$$

were the q's are the coordinates as defined before, and the p's are the conjugate momenta which can be found by

$$p_i = \frac{\partial \mathscr{L}}{\partial \dot{q}_i} \tag{2.10}$$

The resulting momenta are

$$p_x = \dot{x} + \dot{R} - \dot{\theta}\,y \tag{2.11}$$

$$p_y = \dot{y} + \dot{\theta}\,x + \dot{\theta}\,R \tag{2.12}$$

$$p_z = \dot{z} \tag{2.13}$$

After solving the above equations for the $\dot{q}_i$'s and substituting into Eq (2.9), the resulting Hamiltonian is

$$H = \frac{1}{2} p_x^2 + \frac{1}{2} p_y^2 + \frac{1}{2} p_z^2 - \frac{\mu}{R} + \frac{\mu}{R^2} q_x - \frac{\mu}{R^3} q_x^2$$
$$+ \frac{1}{2} \frac{\mu}{R^3} q_y^2 + \frac{1}{2} \frac{\mu}{R^3} q_z^2 - \dot{R} p_x + \dot{\theta} p_x q_y$$
$$- \dot{\theta} q_x p_y - \dot{\theta} R p_y \qquad (2.14)$$

To find Hamilton's equations of motion for the p's and q's the following relationships must be used:

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad (2.15)$$

$$\dot{p}_i = - \frac{\partial H}{\partial q_i} \qquad (2.16)$$

Substituting the Hamiltonian into these two equations yields the following system of equations which describe the motion of body B with respect to body A.

$$\dot{q}_x = p_x - \dot{R} + \dot{\theta} q_y \qquad (2.17)$$

$$\dot{q}_y = p_y - \dot{\theta} q_x - \dot{\theta} R \qquad (2.18)$$

$$\dot{q}_z = p_z \qquad (2.20)$$

and

$$\dot{p}_x = - \frac{\mu}{R^2} + \frac{2\mu}{R^3} q_x + \dot{\theta} p_y \qquad (2.21)$$

$$\dot{p}_y = - \frac{\mu}{R^3} q_y - \dot{\theta} p_x \qquad (2.22)$$

$$\dot{p}_z = -\frac{\mu}{R^3} q_z \tag{2.23}$$

Putting these equations into matrix form yields the following set of equations.

$$\begin{Bmatrix} \dot{q}_x \\ \dot{p}_x \\ \dot{q}_y \\ \dot{p}_y \\ \dot{q}_z \\ \dot{p}_z \end{Bmatrix} = \begin{bmatrix} 0 & 1 & \dot{\theta} & 0 & 0 & 0 \\ \frac{2\mu}{R^3} & 0 & 0 & \dot{\theta} & 0 & 0 \\ -\dot{\theta} & 0 & 0 & 1 & 0 & 0 \\ 0 & -\dot{\theta} & -\frac{\mu}{R^3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -\frac{\mu}{R^3} & 0 \end{bmatrix} \begin{Bmatrix} q_x \\ p_x \\ q_y \\ p_y \\ q_z \\ p_z \end{Bmatrix} + \begin{Bmatrix} \dot{R} \\ -\frac{\mu}{R^2} \\ -\dot{\theta}R \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{2.24}$$

For the case in which body A is in a circular orbit, the above reduces to a linear, constant coefficient system. However, if body A is in an elliptical orbit, R and $\dot{\theta}$ are time dependent. Nevertheless, both R and $\dot{\theta}$ are periodic; therefore the entire Hamiltonian described above is a linear, time-periodic system.

Constant Coefficient Systems

For body A in a circular orbit, Eq (2.24) can be written in the form $\dot{\tilde{x}} = \tilde{A}\,\tilde{x} + \tilde{f}$ where $\tilde{A}$ is a constant matrix, $\tilde{f}$ is a constant vector, and $\tilde{x}$ is a time dependent vector. As with scalar differential equations, $\tilde{x}(t)$ can be solved by finding the particular and homogenous solutions independently and adding the solution

$$\underset{\sim}{x} = \underset{\sim}{x}_h + \underset{\sim}{x}_p \tag{2.25}$$

The easiest of the two solutions is to find $\underset{\sim}{x}_p$. If $\underset{\sim}{x}_p$ is assumed to be a constant vector then $\dot{\underset{\sim}{x}}_p = 0$. Thus Eq (2.24) becomes

$$\underset{\sim}{0} = \underset{\sim}{A}\,\underset{\sim}{x}_p + \underset{\sim}{f} \tag{2.26}$$

and $\underset{\sim}{x}_p$ is easily obtained.

On the other hand, $\underset{\sim}{x}_h$ is not so easy to find. The homogenous differential equation is

$$\frac{d}{dt}(\underset{\sim}{x}_h) = \underset{\sim}{A}\,\underset{\sim}{x}_h \tag{2.27}$$

rearranging

$$\frac{d\underset{\sim}{x}_h}{\underset{\sim}{x}_h} = \underset{\sim}{A}\,dt \tag{2.28}$$

integrating both sides yields

$$\ln\,(\underset{\sim}{x}_h) = \underset{\sim}{A}\,t + \underset{\sim}{c} \tag{2.29}$$

taking the exponent of both sides

$$\underset{\sim}{x}_h = \exp(\underset{\sim}{A}t + \underset{\sim}{c}) = \underset{\sim}{c}\,\exp(\underset{\sim}{A}t) \tag{2.30}$$

where $\underset{\sim}{c}$ is a constant vector related to the initial conditions.

Although this is not a mathematically rigorous argument, it does show how the solution is obtained. The expression $\exp(\underset{\sim}{A}t)$ is similar to the scalar exponential in that the expansion is

$$\exp(\underset{\sim}{A}t) = \underset{\sim}{I} + \underset{\sim}{A}t + \frac{1}{2!}(\underset{\sim}{A}\,t)^2 + \frac{1}{3!}(\underset{\sim}{A}\,t)^3 + \cdots \qquad (2.31)$$

$\underset{\sim}{A}$, however can be put in the form

$$\underset{\sim}{A} = \underset{\sim}{F}\,\underset{\sim}{J}\,\underset{\sim}{F}^{-1} \qquad (2.32)$$

where $\underset{\sim}{F}$ is the matrix of eigenvectors of $\underset{\sim}{A}$ in column form, and $\underset{\sim}{J}$ is a diagonal matrix of the eigenvalues of $\underset{\sim}{A}$. $\underset{\sim}{J}$ is said to be in Jordan normal form.

If it is noted that

$$(\underset{\sim}{F}\,\underset{\sim}{J}\,\underset{\sim}{F}^{-1})^n = (\underset{\sim}{F}\,\underset{\sim}{J}\,\underset{\sim}{F}^{-1})(\underset{\sim}{F}\,\underset{\sim}{J}\,\underset{\sim}{F}^{-1})\cdots = \underset{\sim}{F}\,\underset{\sim}{J}^n\,\underset{\sim}{F}^{-1} \qquad (2.33)$$

then

$$\exp(\underset{\sim}{A}t) = \underset{\sim}{I} + \underset{\sim}{F}\,[\underset{\sim}{J}t]\,\underset{\sim}{F}^{-1} + \underset{\sim}{F}\,[\tfrac{1}{2!}(\underset{\sim}{J}t)^2]\,\underset{\sim}{F}^{-1} + \underset{\sim}{F}\,[\tfrac{1}{3!}(\underset{\sim}{J}t)^3]\,\underset{\sim}{F}^{-1}$$

$$= \underset{\sim}{F}\exp(\underset{\sim}{J}t)\,\underset{\sim}{F}^{-1} \qquad (2.34)$$

setting $\underset{\sim}{\phi}(t) = \underset{\sim}{F}\exp(\underset{\sim}{J}t)\,\underset{\sim}{F}^{-1}$, the total solution to the differential equations of motion is

$$\underset{\sim}{x}(t) = \underset{\sim}{\phi}(t)\,\underset{\sim}{c} + \underset{\sim}{x}_p \qquad (2.35)$$

Therefore, finding the solution to the equations of motion for the case where body A is in a circular orbit can be done by finding the eigenvalues and eigenvectors and solving Eq (2.26).

Floquet Analysis

The solution for a constant coefficient system has been well studied and yields relatively simple results. The system for this problem, however, is one of time-periodic coefficients. The solution for this type of problem was discovered by Floquet in the latter part of the 1800's. The most common uses for Floquet theory is to find the stability of time-periodic systems in celestial dynamics. Few studies, however, deal with finding a solution to the equations of motion.

If we start with the system of time-periodic differential equations

$$\dot{\underset{\sim}{x}} = \underset{\sim}{A}(t)\ \underset{\sim}{x} \tag{2.36}$$

where $\underset{\sim}{x}$ is the state vector, $\dot{\underset{\sim}{x}}$ is the time derivative of the state vector, $\underset{\sim}{A}(t)$ is a periodic matrix with period T. The numerical solution to this problem is only slightly more difficult than the one for constant coefficients in $\underset{\sim}{A}$; however, the process is different so a discussion of the procedure is included.

Since we have a linear system, its solution can be described as

$$\underset{\sim}{x}(t) = \underset{\sim}{\phi}\ (t,0)\ \underset{\sim}{c}(t_o) \tag{2.37}$$

where the state transition matrix, $\underset{\sim}{\phi}$, has the initial conditions

$$\dot{\underset{\sim}{\phi}}(t,0) = \underset{\sim}{A}(t)\, \underset{\sim}{\phi}(t,0) \tag{2.38}$$

$$\underset{\sim}{\phi}(0,0) = \underset{\sim}{I} \tag{2.39}$$

where $\underset{\sim}{I}$ is the identity matrix.

Floquet theory shows that $\underset{\sim}{\phi}$ can be factored into two matrices $\underset{\sim}{F}$ and $\underset{\sim}{J}$, such that

$$\underset{\sim}{\phi}(t,0) = \underset{\sim}{F}(t)\, \exp(\underset{\sim}{J}t)\, \underset{\sim}{F}^{-1}(0) \tag{2.40}$$

The matrix $\underset{\sim}{J}$ is a constant matrix most conveniently put in Jordan normal form. The diagonal entries of $\underset{\sim}{J}$ are the Poincaré exponents which are related to the system eigenvalues. The matrix $\underset{\sim}{F}$ is a time periodic matrix with the same period, T, as the original system.

For the constant coefficient system, $\underset{\sim}{F}$ would just be the eigenvectors of $\underset{\sim}{A}$. The only difference between the constant coefficient system and the periodic system is that $\underset{\sim}{F}$ is periodic in the latter case. Therefore, solving the Floquet problem requires finding the constant matrix $\underset{\sim}{J}$ and the periodic matrix $\underset{\sim}{F}$ over a single period.

The first step in the Floquet analysis is to find $\underset{\sim}{\phi}(T,0)$. This is called the monodromy matrix. The monodromy matrix is usually found by numerically integrating

$$\dot{\underset{\sim}{\phi}}(t,0) = \underset{\sim}{A}(t) \cdot \underset{\sim}{\phi}(t,0) \tag{2.41}$$

over one period. Having $\phi(T,0)$ and knowing $F(T) = F(0)$ results in

$$\phi(T,0) = F(0)\ \exp(Jt)\ F^{-1}(0) \tag{2.42}$$

which can be written as

$$\exp(Jt) = F^{-1}(0)\ \phi(T,0)\ F(0) \tag{2.43}$$

This shows that $F(0)$ is the matrix of eigenvectors for $\phi(T,0)$.

Also, if λ_i are the eigenvalues of the matrix $\phi(T,0)$. The Poincaré exponents, ω_i, are related to the eigenvalues by the following relationship

$$\lambda_i = \exp(\omega_i T) \tag{2.44}$$

Where the ω_i are the diagonal elements of J, or

$$\omega_i = (1/T)\ \ln\ (\lambda_i) \tag{2.45}$$

The stability information for the system is now present. If any of the Poincaré exponents have positive real parts, the system is unstable. Since the stability information is all that is usually needed, this is where most Floquet analyses stop. For this study, however, the solution to the equations of motion are desired; therefore, the analysis must continue.

In order to find the complete solution, $F(t)$ must be found. Since it is periodic, $F(t)$ is only needed over the first period. By substituting Eq (2.38) into Eq (2.37) and rearranging

$$\dot{\underset{\sim}{F}}(t) = \underset{\sim}{A}(t)\ \underset{\sim}{F}(t) - \underset{\sim}{F}(t)\ \underset{\sim}{J} \tag{2.46}$$

where the initial conditions for the equation is just the matrix of eigenvectors. So by integrating this equation over one period the total solution can be found.

The first problem that arises is that $\underset{\sim}{F}(t)$ and $\underset{\sim}{J}$ can be complex, making any real analysis quite difficult. They can, however, be arranged such that both are completely real. $\underset{\sim}{F}$ should be arranged in column vectors, f_i, if the eigenvector is completely real, then f_i will simply be the eigenvector. On the other hand, if there is a pair of complex eigenvectors (they always appear in conjugate pairs), then the columns will be $f_{i\ real}$ and $f_{i\ im}$.

The matrix $\underset{\sim}{J}$ will no longer be in Jordan normal form, but will be in block diagonal form. Real ω_i's will remain the diagonal elements of $\underset{\sim}{J}$, but the complex pairs of ω_i's will be appear in the following diagonal blocks:

$$\begin{bmatrix} Re(\omega) & Im(\omega) \\ -Im(\omega) & Re(\omega) \end{bmatrix} \tag{2.47}$$

The matrix $\exp(\underset{\sim}{J}t)$ is then replace by the diagonal entries $\exp(\omega_i t)$ for the real ω_i, and the diagonal blocks

$$\exp(\omega_i t)\begin{bmatrix} \cos(\mathrm{Im}(\omega_i)t) & -\sin(\mathrm{Im}(\omega_i)t) \\ \sin(\mathrm{Im}(\omega_i)t) & \cos(\mathrm{Im}(\omega_i)t) \end{bmatrix} \tag{2.48}$$

for complex conjugate pairs.

In this problem, it will also be necessary to find $\underset{\sim}{F}^{-1}(t)$ over one period. $\underset{\sim}{F}(t)$ is always invertable; however, an easier method for finding $\underset{\sim}{F}^{-1}(t)$ exists. If the identity $\underset{\sim}{F}\underset{\sim}{F}^{-1} = \underset{\sim}{I}$ is differentiated with respect to time and subsituted from Eq (2.46), the result is

$$\dot{\underset{\sim}{F}}^{-1}(t) = -\underset{\sim}{F}^{-1}(t)\ \underset{\sim}{A}(t) + \underset{\sim}{J}\ \underset{\sim}{F}^{-1}(t) \tag{2.49}$$

Since the problem that is being dealt with is periodic, there will be a pair of repeated eigenvectors along the velocity vector of the two body orbit (Wiesel, 1981). Since the columns of $\underset{\sim}{F}$ must be independent, another eigenvector must be found. If $\underset{\sim}{X}$ is the vector of p_i and q_i of the two body orbit, then the repeated eigenvector, ζ_1 will be

$$\zeta_1 = \frac{d\underset{\sim}{X}}{dt} \tag{2.50}$$

and the extended eigenvector ζ_2 will be

$$\zeta_2 = \frac{d\underset{\sim}{X}}{dE} \tag{2.51}$$

where E is the energy of the orbit.

Now that $\underset{\sim}{F}(0)$ and $\underset{\sim}{J}$ can be formed, Eq (2.46) can now be integrated. By using harmonic analysis, as outlined in Brouwer and Clemence, one can find a closed form expression for the elements of $\underset{\sim}{F}$. Since $\underset{\sim}{F}$ is periodic it can be expressed as

$$F(i,j) = \tfrac{1}{2}\, c_0 + c_1 \cos t + c_2 \cos 2t + \cdots + \tfrac{1}{2}\, c_n \cos nt$$
$$+ s_1 \sin t + s_2 \sin 2t + \cdots + s_n \sin nt \qquad (2.52)$$

where

$$c_k = \frac{1}{n} \sum_{j=0}^{2n-1} F(j\alpha) \cos kj\alpha, \qquad k = 0,1,2,\cdots,n \qquad (2.53)$$

$$s_k = \frac{1}{n} \sum_{j=1}^{2n-1} F(j\alpha) \sin kj\alpha, \qquad k = 0,1,2,\cdots,n - 1 \quad (2.54)$$

and

$$\alpha = 2\pi/2n \qquad (2.55)$$

Therefore, numerically finding n values of $\underset{\sim}{F}$, spaced over equal intervals of time, yields a complete solution for $\underset{\sim}{F}$. Therefore, a complete homogenous solution has been found.

Particular Solution

In order to find the complete solution to the problem a particular solution, in addition the homogenous solution, must be found. Eq (2.24) is in the form

$$\dot{\underset{\sim}{x}} = \underset{\sim}{A}\, \underset{\sim}{x} + \underset{\sim}{f}(t) \qquad (2.56)$$

Now, introducing the modal variables, $\underset{\sim}{\chi}$, as

$$\underset{\sim}{\chi} = \underset{\sim}{F}^{-1}(t)\ \underset{\sim}{x} \tag{2.57}$$

then

$$\dot{\underset{\sim}{\chi}} = \dot{\underset{\sim}{F}}^{-1}(t)\ \underset{\sim}{x} + F^{-1}(t)\ \dot{\underset{\sim}{x}} \tag{2.58}$$

substituting Eq (2.57) into Eq(2.58) yields

$$\dot{\underset{\sim}{\chi}} = -\underset{\sim}{F}^{-1}(t)\underset{\sim}{A}(t)\underset{\sim}{x} + \underset{\sim}{J}\underset{\sim}{F}^{-1}(t)\underset{\sim}{x} + \underset{\sim}{F}^{-1}(t)\underset{\sim}{A}(t)\underset{\sim}{x} + \underset{\sim}{F}^{-1}(t)\underset{\sim}{f}(t) \tag{2.59}$$

Substituting Eq (2.57) into Eq (2.59) and simplifying results in

$$\dot{\underset{\sim}{\chi}} = \underset{\sim}{J}\ \underset{\sim}{\chi} + \underset{\sim}{F}^{-1}(t)\ \underset{\sim}{f}(t) \tag{2.60}$$

The term, $\underset{\sim}{F}^{-1}(t)\ \underset{\sim}{f}(t)$, is periodic; therefore, it can be integrated over one period to find the Fourier coefficients as was shown for $\underset{\sim}{F}(t)$. The solution for $\underset{\sim}{\chi}$ can then be easily integrated by hand since it is just a series of sines and cosines.

Conclusion

The geometry of the problem and the equations have been shown. A brief overview of the methodology for solving the equations of motion was then presented. The following chapters will utilize the procedures discussed in order to find the solution to the problem of relative motion of two satellites in neighboring elliptical orbits.

III. Circular Solution

Introduction

This chapter will deal exclusively with the solution for the problem with body A being in a circular orbit. It will first be solved as an autonomous system, and then the results will be compared with the solution obtained by using a Floquet analysis as outlined in the previous chapter.

Autonomous Solution

The elements of the $\underset{\sim}{A}$ matrix, R, and θ, for the circular solution will all be constant. Using canonical variables, R, μ, and $\dot{\theta}$ will all be identically equal to one. The resulting equations of motion will be

$$\dot{\underset{\sim}{x}} = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix} \underset{\sim}{x} + \begin{Bmatrix} 0 \\ -1 \\ -1 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{3.1}$$

The first step in finding the exact solution to the circular case is to find the homogenous solution for Eq (3.1). As shown in the previous chapter, this is done by finding the eigenvalues and eigenvectors of the $\underset{\sim}{A}$ matrix.

The eigenvalues are found by the equation

$$\det(\lambda \underset{\sim}{I} - \underset{\sim}{A}) = 0 \tag{3.2}$$

The resulting characteristic equation is

$$\lambda^4 + \lambda^2 = 0 \tag{3.3}$$

Therefore, the eigenvalues are

$$\lambda_{1,2} = 0 \tag{3.4}$$

and

$$\lambda_{3,4} = \pm i \tag{3.5}$$

$$\lambda_{5,6} = \pm i \tag{3.6}$$

The eigenvectors are then found by the equation

$$(\lambda_i \underset{\sim}{I} - \underset{\sim}{A})\ \zeta_i = 0 \tag{3.7}$$

For λ_1 the resulting eigenvector is

$$\underset{\sim}{\zeta}_1 = (\ 0,\ 1,\ -1,\ 0,\ 0,\ 0)^T \tag{3.8}$$

However, there is no independent eigenvector cooresponding to λ_2. This vector, termed the generalized vector, can by found using the expression (Reid 1983)

$$(\underset{\sim}{A} - \lambda\ \underset{\sim}{I})\ \underset{\sim}{\zeta}_2 = \underset{\sim}{\zeta}_1 \tag{3.9}$$

This yields

$$\underset{\sim}{\zeta}_2 = \left(\frac{2}{3},\ 1,\ -1,\ -\frac{1}{3},\ 0,\ 0 \right)^T \tag{3.10}$$

The other four eigenvectors occur in complex conjugate pairs. As in the Floquet analysis outlined in chapter 2, the eigenvectors can be broken up into their real and imaginary parts, and J can be put into blocks of sines and cosines.

The pairs of eigenvectors are

$$\underset{\sim}{\zeta}_3 = \left(\frac{1}{2}\,i,\ -\frac{1}{2},\ 1,\ -\frac{1}{2}\,i,\ 0,\ 0 \right)^T \tag{3.11}$$

$$\underset{\sim}{\zeta}_5 = \left(0,\ 0,\ 0,\ 0,\ 1,\ i \right)^T \tag{3.12}$$

Therefore, the following can be written

$$\underset{\sim}{F} = \begin{bmatrix} 0 & 2/3 & 0 & -1/2 & 0 & 0 \\ 1 & 1 & -1/2 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 & 0 \\ 0 & -1/3 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.13}$$

$$\underset{\sim}{F}^{-1} = \begin{bmatrix} -3 & 2 & 1 & -3 & 0 & 0 \\ 3 & 0 & 0 & 3 & 0 & 0 \\ 0 & 2 & 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.14}$$

$$\exp(\underset{\sim}{J}t) = \begin{bmatrix} 1 & t & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos t & \sin t & 0 & 0 \\ 0 & 0 & -\sin t & \cos t & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos t & \sin t \\ 0 & 0 & 0 & 0 & -\sin t & \cos t \end{bmatrix} \quad (3.15)$$

Multiplying matrices

$$\underset{\sim}{F} \exp(\underset{\sim}{J}t) = \begin{bmatrix} 0 & 2/3 & \frac{1}{2}\sin t & -\frac{1}{2}\cos t & 0 & 0 \\ 1 & t+1 & -\frac{1}{2}\cos t & -\frac{1}{2}\sin t & 0 & 0 \\ -1 & -t-1 & \cos t & \sin t & 0 & 0 \\ 0 & -1/3 & -\frac{1}{2}\sin t & -\frac{1}{2}\cos t & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos t & \sin t \\ 0 & 0 & 0 & 0 & -\sin t & \cos t \end{bmatrix} \quad (3.16)$$

and

$$\underset{\sim}{\phi}(t,0) = \underset{\sim}{F} \exp(\underset{\sim}{J}t) \underset{\sim}{F}^{-1} =$$

$$\begin{bmatrix} 2-\cos t & \sin t & \sin t & 2-2\cos t & 0 & 0 \\ 3t-\sin t & 2-\cos t & 1-\cos t & 3t-2\sin t & 0 & 0 \\ -3t+2\sin t & -2+2\cos t & -1+2\cos t & -3t+4\sin t & 0 & 0 \\ -1+\cos t & -\sin t & -\sin t & -1+2\cos t & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos t & \sin t \\ 0 & 0 & 0 & 0 & -\sin t & \cos t \end{bmatrix} \quad (3.17)$$

So the above matrix is the $\underset{\sim}{\phi}$ matrix and is the solution to the homogenous set of equations.

Finding the particular solution requires finding the solution to the following matrix equations

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix} \underset{\sim}{x}_p = \begin{Bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{3.18}$$

Resulting in

$$\underset{\sim}{x}_p = (\ 0,\ 1,\ -1,\ 1,\ 0,\ 0)^T \tag{3.19}$$

Putting the homogeneous and particular solutions together yields an equation in the following form.

$$\underset{\sim}{x}(t) = \underset{\sim}{\phi}(t,t_o)\ \underset{\sim}{c}(t_o) + \underset{\sim}{x}_p \tag{3.20}$$

Therefore, the next step in finding the total solution is to find the constant vector $\underset{\sim}{c}(t_o)$. Evaluating Eq (3.20) at time equal to zero results in

$$\underset{\sim}{x}(0) = \underset{\sim}{\phi}(0,0)\ \underset{\sim}{c}(0) + \underset{\sim}{x}_p \tag{3.21}$$

Since $\underset{\sim}{\phi}(0,0)$ is the identity matrix

$$\underset{\sim}{c}(0) = \underset{\sim}{x}(0) - \underset{\sim}{x}_p \tag{3.22}$$

Recalling Eqs (2.11), (2.12) and (2.13) for body A in a circular orbit yields the following

$$\underset{\sim}{x}(t) = \begin{Bmatrix} x \\ \dot{x} - y \\ y \\ \dot{y} + x + 1 \\ z \\ \dot{z} \end{Bmatrix} \tag{3.23}$$

Evaluating for time equal to zero and substituting into Eq (3.22) yields.

$$\underset{\sim}{c}(0) = \begin{Bmatrix} x_o \\ \dot{x}_o - y_o - 1 \\ y_o + 1 \\ \dot{y}_o + x_o \\ z_o \\ \dot{z}_o \end{Bmatrix} \tag{3.24}$$

Multiplying out Eq (3.20) with the given results for $\underset{\sim}{\phi}(t,0)$, $\underset{\sim}{c}(0)$, and $\underset{\sim}{x}_p$ directly yield the following equations for the position of body B with respect to A in the original coordinate system referred to in Figure 1.

$$x(t) = 4\, x_o - (\, 3\, x_o + 2\, \dot{y}_o) \cos t + \dot{x}_o \sin t + 2\, \dot{y}_o \tag{3.25}$$

$$\begin{aligned} y(t) = {} & (\, 6\, x_o + 4\, \dot{y}_o) \sin t + 2\, \dot{x}_o \cos t \\ & - (\, 6\, x_o + 3\, \dot{y}_o)\, t - 2\, \dot{x}_o + y_o \end{aligned} \tag{3.26}$$

$$z(t) = z_o \cos t + \dot{z}_o \sin t \tag{3.27}$$

The equations of motion that have been derived do not directly give equations for the velocities, or dot terms. Eq (3.20) gives the equations for the momenta terms, and the velocity equations can then be found. The equations for the momenta are

$$
\begin{aligned}
p_x(t) &= \dot{x}(t) - y(t) \\
&= -\,(3\,x_o + 2\,\dot{y}_o)\sin t - \dot{x}_o \cos t \\
&\quad + (6\,x_o + 3\,\dot{y}_o)\,t + 2\,\dot{x}_o - \dot{y}_o
\end{aligned}
\tag{3.28}
$$

$$
\begin{aligned}
p_y(t) &= \dot{y} + x + 1 \\
&= -\,\dot{x}_o \sin t + (3\,x_o + 2\,\dot{y}_o)\cos t \\
&\quad - 2\,x_o - \dot{y}_o + 1
\end{aligned}
\tag{3.29}
$$

$$
p_z(t) = \dot{z}(t) = -\,z_o \sin t + \dot{z}_o \cos t \tag{3.30}
$$

Therefore, the solutions for the velocity components are

$$
\dot{x}(t) = 3\,x_o + 2\,\dot{y}_o + \dot{x}_o \cos t \tag{3.31}
$$

$$
\dot{y}(t) = -\,2x_o \sin t + (6x_o + 4\dot{y}_o)\cos t - 6x_o - 3\dot{y}_o \tag{3.32}
$$

$$
\dot{z}(t) = -\,z_o \sin t + \dot{z}_o \cos t \tag{3.33}
$$

. These equations are the same as derived in Kaplan. Kaplan, however, derived the equations using Newtonian mechanics, whereas the derivation here is using Hamilton's equations.

Circular Solution Using Floquet Theory

The exact solution derived above should yield the same results as Floquet theory. Briefly, Floquet theory consists of the following steps

1. The $\underset{\sim}{\phi}$ matrix will be integrated over one period.
2. The $\underset{\sim}{F}$ and $\underset{\sim}{J}$ matrices will be found at time equal to zero using the eigenvectors and Poincaré exponents evaluated from $\underset{\sim}{\phi}$ at one period.
3. $\underset{\sim}{F}$ will be integrated over one orbit. Since it is periodic, the results are valid for all time.
4. The fourier coefficients for the individual terms in $\underset{\sim}{F}$ are found.
5. The solution for the $\underset{\sim}{\phi}$ matrix for all time is now available and the solution can be found using the initial conditions the same as was done for the Hamiltonian derivation.

Integrating $\underset{\sim}{\phi}$ over one period yields the following matrix

$$\begin{bmatrix} 1.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 \\ 18.849556 & 1.000000 & 0.000000 & 18.849556 & 0.000000 & 0.000000 \\ -18.849556 & 0.000000 & 1.000000 & -18.849556 & 0.000000 & 0.000000 \\ 0.000000 & 0.000000 & 0.000000 & 1.000000 & 0.000000 & 0.000000 \\ 0.000000 & 0.000000 & 0.000000 & 0.000000 & 1.000000 & 0.000000 \\ 0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 & 1.000000 \end{bmatrix}$$

Finding the eigenvalues for this matrix is not extremely difficult, λ_{1-6} are all one. Finding the appropriate eigenvectors is not quite so easy. The first problem that arises is the that IMSL subroutine eigrf can not handle the problem. Due to the six repeated eigenvalues, IMSL only finds three independent eigenvectors. One is a vector along the velocity of body A as expected, and the other two that it finds are for the out of plane motion. They are

$$(0, 1, -1, 0, 0, 0)^T$$
$$(0, 0, 0, 0, 1, 0)^T$$
$$(0, 0, 0, 0, 0, 1)^T$$

It has already been shown that one of the eigenvectors will be

$$\zeta_2 = \frac{d\underset{\sim}{X}}{dE} \tag{3.34}$$

where E is the energy of the orbit and $\underset{\sim}{X}$ is the position vector of body A with respect the center of the Earth evaluated at time equal to zero or any integer multiple of the period. The vector $\underset{\sim}{X}$, in the coordinates being used, is

$$\underset{\sim}{X} = (R_0, 0, 0, v_0, 0, 0) \tag{3.35}$$

where v is the linear velocity of body A. Therefore, the eigenvector will be

$$\zeta = \left(\frac{dR_o}{dE},\ 0,\ 0,\ \frac{dv_o}{dE},\ 0,\ 0 \right) \tag{3.36}$$

First R_o and v_o must be found as a function of E. Starting with

$$E = (v_o^2 / 2) - (\mu / R_o) \tag{3.35}$$

and

$$v_o = (\mu / R_o)^{1/2} \tag{3.36}$$

Substituting Eq (3.36) into Eq (3.35) yields

$$R_o = - \frac{\mu}{2E} \tag{3.37}$$

also

$$v_o = (- 2 E)^{1/2} \tag{3.38}$$

Taking the derivative of Eqs (3.37) and (3.38) with respect to E yields the eigenvector

$$\zeta_2 = \left(\frac{\mu}{2E^2},\ 0,\ 0,\ -(- 2 E)^{-1/2},\ 0,\ 0 \right)^T \tag{3.39}$$

Since canonical units are being used, $a = 1$, $\mu = 1$, and $E = -1/2$. Therefore,

$$\zeta_2 = (2,\ 0,\ 0,\ -1,\ 0,\ 0)^T \tag{3.40}$$

Four of the eigenvectors have now been found; however, two more are still needed. Finding them by hand turns out to be fairly simple due to the number of zeros present in ϕ.

$$\zeta_3 = (\ 0,\ 1,\ 0,\ 0,\ 0,\ 0)^T \tag{3.41}$$

$$\zeta_4 = (\ 1,\ 0,\ 0,\ -1,\ 0,\ 0)^T \tag{3.42}$$

Putting all the eigenvectors together gives the following

$$\underset{\sim}{F}(0) = \begin{bmatrix} 0 & 2 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.43}$$

also

$$\underset{\sim}{F}^{-1}(0) = \begin{bmatrix} 0 & 0 & -1/3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.44}$$

Since all the eigenvalues are equal to one, all the Poincaré exponents are zero. This means that the $\underset{\sim}{J}$ matrix will be zero, except an off diagonal one due to the generalized eigenvector (ζ_2). Therefore,

$$\underset{\sim}{J} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.45}$$

and

$$\exp(\underset{\sim}{J}t) = \begin{bmatrix} 1 & t & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.46}$$

Now everything has been found that is needed to find $\underset{\sim}{F}(t)$. With $\underset{\sim}{F}(0)$ as the initial conditions, $\underset{\sim}{F}$ was integrated over one orbit. It turned out, however, that the chosen eigenvectors were not periodic. Instead of returning to their original values, the following matrix resulted

$$\underset{\sim}{F}(T) = \begin{bmatrix} 0 & 2 & 0 & 1 & 0 & 0 \\ 1 & 12.6 & 1 & 0 & 0 & 0 \\ -1 & -12.6 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.47}$$

Since $\underset{\sim}{F}(0) \neq \underset{\sim}{F}(T)$, this is not a periodic combination of eigenvectors. However, due to the fact that all the Poincaré exponents are zero, any linear combination of the eigenvectors is still an eigenvector. Therefore, some linear combination of the first two eigenvectors must be found to make the set of eigenvectors a periodic one.

Evaluating the time derivative of $\underset{\sim}{F}$ at zero results in the following

$$\dot{\underset{\sim}{F}}(0) = \underset{\sim}{A}\,\underset{\sim}{F}(0) - \underset{\sim}{F}(0)\,\underset{\sim}{J} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & -2 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix} \tag{3.48}$$

What was done to solve this problem was to find a set of eigenvectors that make the time derivatives of the second and third row of the second column equal to zero at time equal to zero.

The first thing that was attempted was to add a multiple of ζ_1 to ζ_2. If

$$\underset{\sim}{F}(0) = \begin{bmatrix} 0 & 2 & 0 & 1 & 0 & 0 \\ 1 & \alpha & 1 & 0 & 0 & 0 \\ -1 & -\alpha & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.50}$$

then

$$
\underset{\sim}{A}\,\underset{\sim}{F}(0) - \underset{\sim}{F}(0)\underset{\sim}{J} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 1 & 0 & 0 \\ 0 & -3 & 0 & -2 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.51}
$$

Adding the two matrices in this manner did not have the desired affect. The next thing that was tried was to just have a multiple of $\underset{\sim}{\zeta}_1$. The eigenvalue matrix at zero will be

$$
\underset{\sim}{F}(0) = \begin{bmatrix} 0 & 2 & 0 & 1 & 0 & 0 \\ \alpha & 0 & 1 & 0 & 0 & 0 \\ -\alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.52}
$$

and

$$
\underset{\sim}{A}\,\underset{\sim}{F}(0) - \underset{\sim}{F}(0)\underset{\sim}{J} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 1 & 0 & 0 \\ 0 & -3 & 0 & -2 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 & 0 \\ 0 & -\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.53}
$$

It is easy to see that if $\alpha = 3$, then $\dot{\underset{\sim}{F}}(0)$ will be zero in the desired positions.

The eigenvector matrix

$$\underset{\sim}{F}(0) = \begin{bmatrix} 0 & 2 & 0 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.54}$$

and

$$\underset{\sim}{F}^{-1}(0) = \begin{bmatrix} 0 & 0 & 0 & -1/3 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.55}$$

was integrated around one orbit and was periodic.

The next step was to find the Fourier coefficients as described in Chapter 2. The results of the harmonic analysis showed that each element of the $\underset{\sim}{F}$ matrix could then be expressed as a function of time in the following manner

$$\underset{\sim}{F}(t) = \begin{bmatrix} 0 & 2 & \sin t & \cos t & 0 & 0 \\ 3 & 0 & 2-\cos t & \sin t & 0 & 0 \\ -3 & 0 & -2+2\cos t & -2\sin t & 0 & 0 \\ 0 & -1 & -\sin t & -\cos t & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos t & \sin t \\ 0 & 0 & 0 & 0 & -\sin t & \cos t \end{bmatrix} \tag{3.56}$$

Note that these numbers are really accurate to ten significant digits. The trailing zeros were left off to make the matrices easier to read.

$$\exp(\underset{\sim}{J}t) = \begin{bmatrix} 1 & t & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad (3.57)$$

Therefore, to find $\underset{\sim}{\phi}(t)$

$$\underset{\sim}{F}(t)\exp(\underset{\sim}{J}t) = \begin{bmatrix} 0 & 2 & \sin t & \cos t & 0 & 0 \\ 3 & 3t & 2-\cos t & \sin t & 0 & 0 \\ -3 & -3t & -2+2\cos t & -2\sin t & 0 & 0 \\ 0 & -1 & -\sin t & -\cos t & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos t & \sin t \\ 0 & 0 & 0 & 0 & -\sin t & \cos t \end{bmatrix} \qquad (3.58)$$

Recalling that

$$\underset{\sim}{F}^{-1}(0) = \begin{bmatrix} 0 & 0 & -1/3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad (3.59)$$

$$\underset{\sim}{\phi}(t,0) = \underset{\sim}{F}(t)\ \exp(\underset{\sim}{J}t)\ \underset{\sim}{F}^{-1}(0) =$$

$$\begin{bmatrix} 2-\cos t & \sin t & \sin t & 2-2\cos t & 0 & 0 \\ 3t-\sin t & 2-\cos t & 1-\cos t & 3t-2\sin t & 0 & 0 \\ -3t+2\sin t & -2+2\cos t & -1+2\cos t & -3t+4\sin t & 0 & 0 \\ -1+\cos t & -\sin t & -\sin t & -1+2\cos t & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos t & \sin t \\ 0 & 0 & 0 & 0 & -\sin t & \cos t \end{bmatrix} \tag{3.60}$$

As expected, this $\underset{\sim}{\phi}(t,0)$ is identical (within ten significant digits) to the one for the exact solution. From here on the total solution can be found from Eq (3.20) in the same manner as the exact solution. Due to redundancy this will not be shown. However, the total solution will result in the same solution because $\underset{\sim}{\phi}(t,0)$ is the same for both approaches.

Conclusion

The solution to the equations of motion for the circular case are not new. The main purpose for finding this solution was to help verify the computer programs that were used. Since the two solutions are identical, it shows that the numerical approach that was used was correct for the case of a circular orbit.

IV. Elliptical Solution

Introduction

The procedure for finding a numerical solution to the elliptical problem is identical to finding the numerical solution for the circular case. The goal, however, is to find the relative motion solution as a function of eccentricity. This will be done by finding the solution for slightly eccentric orbits. For very small eccentricities (ie. $e = 10^{-7}$) only the linear terms of e will appear. After finding the dependence on e, larger values for the eccentricities will be used to find the equations' dependence on e^2 and e^3.

Homogenous Solution to the Equations of Motion

$\underset{\sim}{\phi}(T,0)$ is found in the same manner as for the circular case. $\dot{\underset{\sim}{\phi}} = \underset{\sim}{A}\,\underset{\sim}{\phi}$ is numerically integrated over one period. For all eccentricities, $\underset{\sim}{\phi}(T,0)$ appears in the following form

$$\underset{\sim}{\Phi}(T,0) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ A & 1 & 0 & B & 0 & 0 \\ C & 0 & 1 & D & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad (4.1)$$

Where A,B,C,D are some value.

The eigenvalues, as in the circular case, are all of magnitude equal to one; therefore, all of the Poincaré exponents are zero.

As with the circular case, IMSL cannot find distinct eigenvectors to the matrix; however, it is not too difficult to find the eigenvectors by hand due to the abundance of zeros in the matrix.

$$(\lambda \underset{\sim}{I} - \underset{\sim}{A}) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ A & 0 & 0 & B & 0 & 0 \\ C & 0 & 0 & D & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad (4.2)$$

Four of the eigenvectors are easy to determine. They are

$$\underset{\sim}{\zeta}_3 = (\ 0,\ 1,\ 0,\ 0,\ 0,\ 0)^T \qquad (4.3)$$

$$\underset{\sim}{\zeta}_4 = (\ 1,\ 0,\ 0,\ -A/B,\ 0,\ 0)^T \qquad (4.4)$$

$$\underset{\sim}{\zeta}_5 = (\ 0,\ 0,\ 0,\ 0,\ 1,\ 0)^T \qquad (4.5)$$

$$\underset{\sim}{\zeta}_6 = (\ 0,\ 0,\ 0,\ 0,\ 0,\ 1)^T \qquad (4.6)$$

and, as shown in Chapter 2, there is a repeated eigenvector and its corresponding extended eigenvector. Recalling

$$\zeta_1 = \frac{d\tilde{X}}{dt} \tag{4.7}$$

and

$$\zeta_2 = \frac{d\tilde{X}}{dE} \tag{4.8}$$

the last two eigenvectors can be found. The repeated vector will be

$$\zeta_1 = (\ 0,\ \mu/R_o^2\ ,\ v_o,\ 0,\ 0,\ 0) \tag{4.9}$$

Where the zero subscript denotes time equal to zero. Using canonical units

$$\mu/R^2 = 1/(1-e)^2 \tag{4.10}$$

From Kaplan (1976)

$$e = (R_o v_o^2\ /\ \mu) - 1 \tag{4.11}$$

Which leads to

$$v_o = [(\mu/R_o)\ (e + 1)]^{1/2} \tag{4.12}$$

thus

$$v_o = [(e + 1)/(e - 1)]^{1/2} \tag{4.13}$$

Now only the second eigenvector needs to be found. Recalling Eq (3.35)

$$E = (v_o^2 / 2) - (\mu / R_o) \tag{4.14}$$

Substituting for v_o and solving for R_o results in

$$R_o = (e - 1)/2E \tag{4.15}$$

thus

$$dR_o/dE = \frac{1 - e}{2 E^2} \tag{4.16}$$

However, for canonical units the semimajor axis, a, will be equal to one, and $E = -1/2$. Thus,

$$dR_o/dE = 2 \ (1 - e) \tag{4.17}$$

Now, in order to find dv_o/dE, start with

$$v_o = [(\mu/R_o) \ (e + 1)]^{1/2} \tag{4.18}$$

Using the chain rule

$$dv_o/dE = (\delta v_o/\delta R_o) \ (\delta R_o/dE)$$

$$= - \ (1/4) \ [(1 + e)/R_o^3]^{1/2} \ (1 - e) \ / \ 2E^2 \tag{4.19}$$

simplifying,

$$dv_o/dE = - \ [(1 + e)/(1 - e)]^{1/2} \tag{4.20}$$

Since the eigenvectors have all been found, the matrix $\underset{\sim}{F}(0)$ will be

$$\underset{\sim}{F}(0) = \begin{bmatrix} 0 & 2(1-e) & 0 & 1 & 0 & 0 \\ \dfrac{1}{(1-e)^2} & 0 & 1 & 0 & 0 & 0 \\ \left[\dfrac{1+e}{1-e}\right]^{1/2} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\left[\dfrac{1+e}{1-e}\right]^{1/2} & 0 & -A/B & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.21}$$

The next step is to integrate $\underset{\sim}{F}$ over one period. This was done to the matrix above and, as in the circular case, it was not periodic. The method for making $\underset{\sim}{F}$ periodic for the circular solution was to make the second column of the second and third rows of $\dot{\underset{\sim}{F}}$ equal to zero by using a multiple of the first eigenvalue. However, there is no constant that will make this happen. Nevertheless, if the first eigenvector is multiplied by three as before, then

$$\underset{\sim}{F}(0) = \begin{bmatrix} 0 & 2(1-e) & 0 & 1 & 0 & 0 \\ \dfrac{3}{(1-e)^2} & 0 & 1 & 0 & 0 & 0 \\ 3\left[\dfrac{1+e}{1-e}\right]^{1/2} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\left[\dfrac{1+e}{1-e}\right]^{1/2} & 0 & -A/B & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.22}$$

$$\dot{\underset{\sim}{F}}(3,2) = 0 \tag{4.23}$$

and

$$\dot{\underset{\sim}{F}}(2,2) = -e/(1-e)^2 \tag{4.34}$$

For the case where $e = 0$, Eq (4.34) reduces to zero.

This method was tried and resulted in a periodic function for $\underset{\sim}{F}(t)$. Fifty values of $\underset{\sim}{F}$ were saved along equal increments of time during one period and the Fourier coefficients were found as discussed in Chapter 2.

Table 4.1 lists the eccentricities and the calculated values for the Fourier coefficients of the first column of the first row of $\underset{\sim}{F}(t)$.

Table 4.1
Fourier coefficients for F(1,1)

e	s_1	s_2	s_3
.0000001	-.000000300	*	*
.0000010	-.000003000	*	*
.0000100	-.000030000	*	*
.0001000	-.000300000	*	*
.0005000	-.001500000	-.000000750	*
.0008000	-.002399999	-.000001920	-.000000002
.0010000	-.002999999	-.000003000	-.000000003
.0050000	-.014999859	-.000074999	-.000000422
.0080000	-.023999424	-.000191992	-.000001728
.0100000	-.029998874	-.000299980	-.000003375
.0200000	-.059990999	-.001199680	-.000026990
.0300000	-.089969627	-.002698380	-.000091048
$s_1 = -3e + 1.125e^2$ $s_2 = -3e^2$ $s_3 = -3.375e^3$			

where the stars indicate zero to nine significant digits.

For $e = 10^{-7}$, one can see the linear dependence on e. As the eccentricity increases, the square and cubic are no longer negligible and can be extracted from the data. The resulting equation for $\underset{\sim}{F}(1,1)$ as a function of both eccentricity and time is

$$\underset{\sim}{F}(1,1) = (-3e + 1.125e^2) \sin t - 3e^2 \sin 2t - 3.375e^3 \sin 3t \tag{4.35}$$

The rest of the data to determine the coefficients for the other elements of $\underset{\sim}{F}(t)$ is in Appendix B. The resulting equations for the rest of the elements of $\underset{\sim}{F}$ are as follows:

$$\underset{\sim}{F}(1,2) = 2 + e^2 + [-2e + (3/4)e^3] \cos t - e^2 \cos 2t - (3/4)e^3 \cos 3t$$

$$\underset{\sim}{F}(1,3) = [1 - 2e + (5/8)e^2 + .75e^3] \sin t + [e - 2e^2 + (1/3)e^3] \sin 2t + [(9/8)e^2 - 2.25e^3] \sin 3t + (4/3)e^3 \sin 4t$$

$$\underset{\sim}{F}(1,4) = - e + [1 - (9/8)e^2] \cos t + [e - (4/3)e^3) \cos 2t + (9/8)e^2 \cos 3t - (4/3) \cos 4t$$

$$\underset{\sim}{F}(1,5) = F(1,6) = 0$$

$$\underset{\sim}{F}(2,1) = 3 + 1.5e^2 + (6e + 2.25e^3) \cos t + 7.5e^2 \cos 2t + 9.75e^3 \cos 3t$$

$$\underset{\sim}{F}(2,2) = (-e + .38e^3) \sin t - e^2 \sin 2t - (9/8)e^3 \sin 3t$$

$$\underset{\sim}{F}(2,3) = 2 - 2.5e + e^2 - .87e^3 + (-1 + 4e + 3.875e^2 - 1.5e^3)\cos t + (-1.5e + 5e^2 + 4.16e^3)\cos 2t - (2.12e^2 + 5.8e^3)\cos 3t + 3e^3 \cos 4t$$

$$\underset{\sim}{F}(2,4) = [1 - (3/8)e^2]\sin t + (1.5e - e^3)\sin 2t + 2.215e^2 \sin 3t - 2.96e^3 \sin 4t$$

$$\underset{\sim}{F}(2,5) = \underset{\sim}{F}(2,6) = 0$$

$$\underset{\sim}{F}(3,1) = -3 + 1.5e^2 + (-3e + 1.875e^3)\cos t - 3e^2 \cos 2t - 3.375e^3 \cos 3t$$

$$\underset{\sim}{F}(3,2) = 0$$

$$\underset{\sim}{F}(3,3) = -2 + .5e + e^2 - .25e^3 + (2 - 2e - .5e^2 + 1.244e^3)\cos t + (1.5e - 2e^2 - .4e^3)\cos 2t + (1.5e^2 - 2.25e^3)\cos 3t$$

$$\underset{\sim}{F}(3,4) = (-2 + .5e^2)\sin t + (-1.5e + 1.08e^3)\sin 2t - 1.5e^2 \sin 3t - (5/3)e^3 \sin 4t$$

$$\underset{\sim}{F}(3,5) = F(3,6) = F(4,1) = 0$$

$$\underset{\sim}{F}(4,2) = -1 + .5e^2 + [-e + (5/8)e^3]\cos t - e^2 \cos 2t + (9/8)e^3 \cos 3t$$

$$\underset{\sim}{F}(4,3) = [-1 + (7/8)e^2]\sin t + [-e + (7/6)e^3]\sin 2t - (9/8)e^2 \sin 3t + (4.3)e^3 \sin 4t$$

$$\underset{\sim}{F}(4,4) = [-1 - (3/8)e^2)\cos t + [-e - (1/6)e^3]\cos 2t - (9/8)e^2 \cos 3t - (4/3)e^3 \cos 4t$$

$$\underset{\sim}{F}(4,5) = \underset{\sim}{F}(4,6) = \underset{\sim}{F}(5,1) = \underset{\sim}{F}(5,2) = \underset{\sim}{F}(5,3) = \underset{\sim}{F}(5,4) = 0$$

$$\underset{\sim}{F}(5,5) = -1.5e - 1.5e^2 - 1.5e^3 + [1 + e + (5/8)e^2 + (5/8)e^3] \cos t + (.5e + .5e^2 + .168e^3) \cos 2t + [(3/8)e^2 + (3/8)e^3] \cos 3t + (1/3)e^3 \cos 4t$$

$$\underset{\sim}{F}(5,6) = [1 - e - (1/8)e^2 + (1/8)e^3] \sin t + (.5e - .5e^2 - .165e^3) \sin 2t + [(3/8)e^2 - (3/8)e^3] \sin 3t + (1/3)e^3 \sin 4t$$

$$\underset{\sim}{F}(6,1) = F(6,2) = F(6,3) = F(6,4) = 0$$

$$\underset{\sim}{F}(6,5) = [-1 - e - (5/8)e^2 - (5/8)e^3] \sin t + [-e - e^2 -(1/3)e^3] \sin 2t + (-1.125e^2 - 1.125e^3) \sin 3t - 1.35e^3 \sin 4t$$

$$\underset{\sim}{F}(6,6) = 1 - e - (1/8)e^2 + (1/8)e^3 + [e - e^2 - (1/3)e^3] \cos t + [(9/8)e^2 + (9/8)e^3] \cos 3t + (4/3)e^3 \cos 4t \qquad (4.36)$$

Since $\underset{\sim}{F}(t)$ has been found, finding $\phi(t,0)$ is now relatively simple. As in the circular solution

$$\exp(\underset{\sim}{J}t) = \begin{bmatrix} 1 & t & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad (4.37)$$

Finding the solution for $\underset{\sim}{F}^{-1}(0)$ is tedious, but relatively simple. It will be a matrix in the form

$$\underset{\sim}{F}^{-1}(t) = \begin{bmatrix} 0 & 0 & 1/b & 0 & 0 & 0 \\ (1-f)/c & 1 & 0 & -g/c & 0 & 0 \\ 0 & 1 & -a/b & 0 & 0 & 0 \\ f & 0 & 0 & g & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.38}$$

where

$$a = 3/(1 - e)^2$$

$$b = -3\,[(1 + e)/(1 - e)]^{1/2}$$

$$c = 2\,(1 - e)$$

$$d = -\,[(1 + e)/(1 - e)]^{1/2}$$

$$e = -\,A/B \text{ (from Eq (4.1))}$$

$$f = -d/(ec - d)$$

$$g = c/(ec - d) \tag{4.39}$$

The total solution for $\underset{\sim}{\phi}(t,0)$ can now be expressed in terms of eccentricity cubed by the following matrix multiplication

$$\underset{\sim}{\phi}(t,0) = \underset{\sim}{F}(t)\ \exp(\underset{\sim}{J}t)\ \underset{\sim}{F}^{-1}(0) \tag{4.40}$$

Combining Eqs (4.36), (4.37), and (4.38) the homogenous solution for $\dot{\underset{\sim}{x}} = \underset{\sim}{A}(t)\ \underset{\sim}{x}$ is now determined as an expansion of eccentricity in Eq (4.40).

Particular solution

The complete solution for the relative motion is almost complete. All that is needed is the particular solution. Recalling Eq (2.59)

$$\dot{\chi} = \underset{\sim}{J}\,\chi + \underset{\sim}{F}^{-1}(t)\,\underset{\sim}{f}(t) \qquad (4.41)$$

Since $\underset{\sim}{F}^{-1}(t)\,\underset{\sim}{f}(t)$ is periodic it was integrated and the Fourier coefficients were found as done for $\underset{\sim}{F}(t)$. The solution for χ is then easy to integrate by hand and results in the following:

$$\begin{aligned}
\chi(1) = K_1 + K_2 &+ [-1 + (2/3)e - (1/3)e^2 + .5e^3]\ t \\
&+ [(4/3) - (8/3) + (7/6)e^2 + (1/3)e^3] \sin t \\
&+ [e - (11/6)e^2 + .4125e^3] \sin 2t \\
&+ [(9.43)e^2 - 1.5e^3] \sin 3t + e^3 \sin 4t
\end{aligned}$$

$$\chi(2) = K_2$$

$$\begin{aligned}
\chi(3) = K_3 &+ (-e - .75e^2)\ t + (-2 - 1.75e^2) \sin t \\
&+ [-(3/2)e - (2/3)e^3] \sin 2t - (17/12)e^2 \sin 3t \\
&\qquad - 1.48e^3 \sin 4t
\end{aligned}$$

$$\begin{aligned}
\chi(4) = K_4 &- (2 - 2.75e^2) \cos t - [(3/2)e - 2.4125e^3) \cos 2t \\
&\qquad - (17/12)e^2 \cos 3t - 1.48e^3 \cos 4t
\end{aligned}$$

$$\chi(5) = \chi(6) = 0 \qquad (4.42)$$

Since the particular solution can be any function that satisfies the differential equation, any initial value for χ

will be valid; however, $\chi(0) = 0$ leads to easier expressions. This can be done by setting $K_{1-3} = 0$ and

$$K_4 = 2 + (3/2)e + (4/3)e^2 + 0.9325e^3 \qquad (4.43)$$

Now $\chi(t)$ has been found, the particular solution is

$$\underset{\sim}{x}_p = \underset{\sim}{F}(t)\ \chi(t) \qquad (4.44)$$

where $\underset{\sim}{F}(t)$ is expressed in Eq (4.36) and $\chi(t)$ in Eq(4.42)

Complete solution

Recalling Eq (2.35),

$$\underset{\sim}{x}(t) = \phi(t,0)\ \underset{\sim}{c} + \underset{\sim}{x}_p \qquad (4.45)$$

however, since $\chi(0) = 0$

$$\underset{\sim}{c} = \underset{\sim}{x}_o \qquad (4.46)$$

and Eq (4.45) can be written

$$\underset{\sim}{x}(t) = \phi(t,0)\ \underset{\sim}{x}_o + \underset{\sim}{x}_p \qquad (4.47)$$

where $\phi(t,0)$ is expressed in Eq (4.40), $\underset{\sim}{x}_o$ in Eq (3.23), and x_p in Eq (4.44)

Conclusion

Using the expressions derived in this chapter, the solution to the equations of motion can be expressed as a function of eccentricity and time. The results reduce to the circular solution, but the actual equations for the elliptical solution are rather complicated to write out in an expanded form.

V. Conclusion

An expression for the solution to the relative motion problem has been found. Although this solution is limited to small eccentricities (the error is approximately e^3), it has wide applications. It could be applied to traffic management around the space station, Shuttle activities, or any other application that utilizes orbits of small eccentricities.

These equations of motion are an improvement over previous methods. They are not limited to body A being in a circular orbit as in some previous studies. The results are valid even at very close distances, which is a weakness of the methods based on subtracting the two position vectors.

This analysis also has the advantage of being able to calculate the necessary changes in velocity for rendezvous. If we recall the basic solution to the equation of motion

$$\underset{\sim}{x}(t) = \underset{\sim}{\phi}(t,0)\,\underset{\sim}{x}_o + \underset{\sim}{x}_p \tag{5.1}$$

The values of $\underset{\sim}{\phi}(t,0)$, $\underset{\sim}{x}_o$, and $\underset{\sim}{x}_p$ have previously been found. For the rendezvous problem, the initial and final times are known. Also, the initial position is known, and the final position is simply zero for the three different directions. The only values that are not known are the initial and final velocities. This results in a system of six equations and six unknowns. These are not difficult to solve since all the other values are constants.

One difficulty in using this solution occurs when the desired initial time is not at perigee, but at some time, t_o. This problem can be overcome by using the fact that

$$\phi(t,0) = \phi(t,t_o)\ \phi(t_o,0) \tag{5.2}$$

Finding the required solution, therefore, requires solving Eq (5.2).

There are two limitations to this solution. The first is that the relative distances between the two bodies must be small. The second limitation is that the solution is for small eccentricities. Equations for greater eccentricities can be found using the same approach as this thesis. This could be done with the same computer code, finding the dependence on higher orders of eccentricity.

Utilization of these equations yield themselves best to computer analysis since they are relatively lengthy expressions. The equations being somewhat unwieldy are only a minor inconvenience. One must remember, however, that the equations are in canonical units.

Appendix A: Explanation of Computer Code

Introduction

This appendix is meant to describe the code that was used for this thesis. Hopefully, this will make it easier to understand the numerical processes used.

Computer Code

The main programs were relatively simple. Their purpose was to initialize some of the parameters, mainly eccentricity, and call the numerical integrator. There were actually three main programs. The first integrated the $\underset{\sim}{\phi}$ matrix over one orbit and its output was just $\underset{\sim}{\phi}(T,0)$. The second integrated the $\underset{\sim}{F}$ matrix over one period and its output was the values of the $\underset{\sim}{F}$ matrix for fifty evenly spaced time intervals over one period. The last integrated $\underset{\sim}{F}^{-1}$ over one period and the output was fifty evenly spaced values of $\underset{\sim}{F}^{-1}\ \underset{\sim}{f}(t)$, where $\underset{\sim}{f}(t)$ is the forcing function.

The numeric integrator that was used was Haming. It is a fourth order predictor-corrector capable of integrating systems of first order differential equations. Using it entails forming a subroutine called "rhs," where rhs calculates the right hand side of the equations of motion.

The position of body A was necessary in the subroutine rhs, so it was calculated using a Newton-Raphson method to find Kepler's equation.

The fourier coefficients were found using the approach defined in chapter 2.

Appendix B: Data From Harmonic Analyses

Table B.1
Fourier Coefficients for F(1,2)

e	c_0	c_1	c_2	c_3
.0000001	2.00000000	-.000000200	*	*
.0000010	2.00000000	-.000002000	*	*
.0000100	2.00000000	-.000020000	*	*
.0001000	2.00000001	-.000200000	-.000000010	*
.0005000	2.00000025	-.001000000	-.000000250	*
.0008000	2.00000064	-.001600000	-.000000640	*
.0010000	2.00000100	-.002000000	-.000001000	*
.0050000	2.00002500	-.009999906	-.000025000	-.000000094
.0080000	2.00006400	-.015999616	-.000063997	-.000000384
.0100000	2.00010000	-.019999249	-.000099994	-.000000750
.0200000	2.00040000	-.039994000	-.000399894	-.000005998
.0300000	2.00090000	-.059979751	-.000899460	-.000020233

$c_0 = 2 + e^2$
$c_1 = -2e + (3/4)e^3$
$c_2 = -e^2$
$c_3 = -(3/4)e^3$

Table B.2
Fourier Coefficients for F(1,3)

e	s_1	s_2	s_3	s_4
.0000001	.999999800	.000000010	*	*
.0000010	.999998000	.000000100	*	*
.0000100	.999980000	.000001000	*	*
.0001000	.999800006	.000099980	.000000011	*
.0005000	.999000016	.000499500	.000000281	*
.0008000	.998400400	.000798720	.000000719	*
.0010000	.998000626	.000998000	.000001123	.000000001
.0050000	.990015719	.004950042	.000027844	.000000165
.0080000	.984040383	.007872176	.000070848	.000000672
.0100000	.980063246	.009800347	.000110251	.000001307
.0200000	.960255944	.019202878	.000432018	.000010239
.0300000	.940582466	.028210067	.000951858	.000033836

$s_1 = 1 - 2e + (5/8)\ e^2 + (3/4)e^3$
$s_2 = e - 2e^2 + (1/3)e^3$
$s_3 = (9/8)e^2 - 2.25\ e^3$
$s_4 = (4/3)e^3$

Table B.3
Fourier Coefficient for F(1,4)

e	c_0	c_1	c_2	c_3	c_4
.0000001	-.000000100	1.0000000	.000000100	*	*
.0000010	-.000001000	1.0000000	.000001000	*	*
.0000100	-.000010000	1.0000000	.000010000	*	*
.0001000	-.000100000	1.0000000	.000100000	.000000011	*
.0005000	-.000500000	0.9999997	.000500000	.000000281	*
.0008000	-.000800000	0.9999993	.000800000	.000000720	*
.0010000	-.001000000	0.9999989	.000999999	.000001125	.000000001
.0050000	-.005000000	0.9999719	.004999833	.000028124	.000000167
.0080000	-.008000000	0.9999280	.007999317	.000071993	.000000683
.0100000	-.010000000	0.9998875	.009998667	.000112482	.000001333
.0200000	-.020000000	0.9995502	.019989335	.000449719	.000010659
.0300000	-.030000000	0.9989876	.029964009	.001011077	.000035942

$c_0 = -e$

$c_1 = 1 - 1.125e^2$

$c_2 = e + (4/3)e^2$

$c_3 = 1.125e^2$

$c_4 = -(4/3)e^3$

Table B.4
Fourier Coefficient for F(2,1)

e	c_0	c_1	c_2	c_3
.0000001	3.00000000	.00000060	*	*
.0000010	3.00000000	.00000600	*	*
.0000100	3.00000000	.00006000	*	*
.0001000	3.00000002	.00060000	.000000075	*
.0005000	3.00000038	.00300000	.000001875	.000000001
.0008000	3.00000096	.00480000	.000004800	.000000005
.0010000	3.00000150	.00600000	.000007500	.000000010
.0050000	3.00003750	.03000028	.000187500	.000001219
.0080000	3.00009600	.04800115	.000480004	.000004992
.0100000	3.00015001	.06000225	.000750010	.000009750
.0200000	3.00060018	.12001800	.003000160	.000077996
.0300000	3.00135091	.18006080	.006750811	.000263222

$c_0 = 3 + 1.5e^2$

$c_1 = 6e + 2.25\ e^3$

$c_2 = 7.5e^2$

$c_3 = 9.75e^3$

Table B.5
Fourier Coefficients for F(2,2)

e	s_1	s_2	s_3
.0000001	-.000000100	*	*
.0000010	-.000001000	*	*
.0000100	-.000010000	*	*
.0001000	-.000100000	-.000000010	*
.0005000	-.000500000	-.000000250	*
.0008000	-.000800000	-.000000640	*
.0010000	-.001000000	-.000001000	-.000000001
.0050000	-.004999953	-.000025000	-.000000141
.0080000	-.007999808	-.000063997	-.000000576
.0100000	-.009999620	-.000099992	-.000001124
.0200000	-.019996997	-.000399894	-.000008997
.0300000	-.029989877	-.000899461	-.000030350

$s_1 = -e + .38e^3$

$s_2 = -e^2$

$s_3 = -1.125e^3$

Table B.6
Fourier Coefficient for F(2,3)

e	c_0	c_1	c_2	c_3	c_4
.0000001	1.99999975	-.999999600	-.000000150	*	*
.0000010	1.99999750	-.999996000	-.000001500	*	*
.0000100	1.99997500	-.999960000	-.000015000	*	*
.0001000	1.99975001	-.999600039	-.000149950	-.000000021	*
.0005000	1.99875025	-.998000969	-.000748751	-.000000530	*
.0008000	1.99800064	-.996802479	-.001196802	-.000001357	-.000000002
.0010000	1.99750100	-.996003874	-.001495004	-.000002119	-.000000003
.0050000	1.98752489	-.980096688	-.007375520	-.000052315	-.000000364
.0080000	1.98006356	-.968247237	-.011682131	-.000132691	-.000001480
.0100000	1.97519913	-.960386013	-.014504160	-.000206046	-.000002873
.0200000	1.95039312	-.921538203	-.028033229	-.000798741	-.000022310
.0300000	1.92587697	-.883448018	-.040611974	-.001740759	-.000073048

$c_0 = 2 - 2.5e + e^2 - .87e^3$

$c_1 = -1 + 4e + 3.875e^2 + 1.5e^3$

$c_2 = -1.5e + 5e^2 + 4.16e^3$

$c_3 = 2.12e^2 + 5.8e^3$

$c_4 = -3e^3$

Table B.7
Fourier Coefficient for F(2,4)

e	s_0	s_1	s_2	s_3
.0000001	1.000000000	.000000150	*	*
.0000010	1.000000000	.000001500	*	*
.0000100	1.000000000	.000050000	*	*
.0001000	.999999996	.000150000	.000000021	*
.0005000	.999999906	.000750000	.000000531	*
.0008000	.999999760	.001200000	.000001360	.000000002
.0010000	.999999625	.001499999	.000002125	.000000003
.0050000	.999990625	.007499885	.000053129	.000000370
.0080000	.999976000	.011999531	.000135993	.000001515
.0100000	.999962500	.014999084	.000212482	.000002958
.0200000	.999849996	.029992666	.000849711	.000023656
.0300000	.999662488	.044975252	.001911039	.000079808

$s_1 = 1 - .375e^2$

$s_2 = 1.5e - e^3$

$s_3 = 2.125e^2$

$s_4 = 2.96e^3$

Table B.8
Fourier Coefficient for F(3,1)

e	c_0	c_1	c_2	c_3
.0000001	-3.00000000	-.000000300	*	*
.0000010	-3.00000000	-.000003000	*	*
.0000100	-3.00000000	-.000030000	*	*
.0001000	-2.99999999	-.000300000	-.000000030	*
.0005000	-2.99999963	-.001500000	-.000000750	*
.0008000	-2.99999904	-.002400000	-.000001920	-.000000002
.0010000	-2.99999850	-.002999998	-.000003000	-.000000003
.0050000	-2.99996250	-.014999766	-.000074998	-.000000422
.0080000	-2.99990400	-.023999040	-.000191990	-.000001728
.0100000	-2.99985000	-.029998124	-.000299975	-.000003375
.0200000	-2.99939994	-.059984996	-.001199600	-.000026989
.0300000	-2.99864970	-.089949371	-.002697975	-.000091037

$c_0 = -3 + 1.5e^2$

$c_1 = -3e + 1.875e^3$

$c_2 = -3e^2$

$c_3 = -3.375e^3$

Table B.9
Fourier Coefficient for F(3,3)

e	c_0	c_1	c_2	c_3	c_4
.0000001	-1.99999995	1.99999980	.000000150	*	*
.0000010	-1.99999950	1.99999800	.000001500	*	*
.0000100	-1.99999500	1.99998000	.000015000	*	*
.0001000	-1.99994999	1.99980000	.000149980	.000000015	*
.0005000	-1.99974975	1.99899988	.000749500	.000000375	*
.0008000	-1.99959936	1.99839968	.001198720	.000000959	*
.0010000	-1.99949900	1.99799950	.001498000	.000001498	.000000002
.0050000	-1.99747503	1.98998766	.007449949	.000037218	.000000207
.0080000	-1.99593613	1.98396864	.011871793	.000094846	.000000842
.0100000	-1.99490025	1.97995124	.014799600	.000147744	.000001640
.0200000	-1.98960196	1.95980991	.029196931	.000581910	.000012904
.0300000	-1.98410655	1.93958327	.043190086	.001288814	.000042819

$c_0 = -2 + .5e + e^2 - .25e^3$

$c_1 = 2 - 2e - .5e^2 + 1.244e^3$

$c_2 = 1.5e - 2e^2 - .4e^3$

$c_3 = 1.5e^2 - 2.25e^3$

$c_4 = (5/3)e^3$

Table B.10
Fourier Coefficient for F(3,4)

e	s_1	s_2	s_3	s_4
.0000001	-2.00000000	-.000000150	*	*
.0000010	-2.00000000	-.000001500	*	*
.0000100	-2.00000000	-.000015000	*	*
.0001000	-2.00000000	-.000150000	-.000000015	*
.0005000	-1.99999988	-.000750000	-.000000375	*
.0008000	-1.99999968	-.001199999	-.000000960	*
.0010000	-1.99999950	-.001499999	-.000001500	-.000000002
.0050000	-1.99998750	-.007499865	-.000037499	-.000000208
.0080000	-1.99996800	-.011999445	-.000095993	-.000000853
.0100000	-1.99995000	-.014998917	-.000149984	-.000001667
.0200000	-1.99980006	-.029991334	-.000599737	-.000013326
.0300000	-1.99955033	-.044970761	-.001348672	-.000044943

$s_1 = -2 + .5e^2$

$s_2 = -1.5e + 1.08e^3$

$s_3 = -1.5e^2$

$s_4 = -(5/3)e^3$

Table B.11
Fourier Coefficient for F(4,2)

e	c_0	c_1	c_2	c_3
.0000001	-1.00000000	-.000000100	*	*
.0000010	-1.00000000	-.000001000	*	*
.0000100	-1.00000000	-.000010000	*	*
.0001000	-0.99999999	-.000100000	-.000000010	*
.0005000	-0.99999988	-.000500000	-.000000250	*
.0008000	-0.99999968	-.000800000	-.000000640	*
.0010000	-0.99999950	-.000999999	-.000001000	-.000000001
.0050000	-0.99998750	-.004999922	-.000024999	-.000000141
.0080000	-0.99996800	-.007999680	-.000063997	-.000000576
.0100000	-0.99995000	-.009999376	-.000099991	-.000001125
.0200000	-0.99979998	-.019995000	-.000399866	-.000008996
.0300000	-0.99954990	-.029983123	-.000899325	-.000030346

$c_0 = -1 + .5e^2$

$c_1 = -e + .625e^3$

$c_2 = -e^2$

$c_3 = 1.125e^3$

Table B.12
Fourier Coefficient for F(4,3)

e	s_1	s_2	s_3	s_4
.0000001	-1.00000000	-.000000100	*	*
.0000010	-1.00000000	-.000001000	*	*
.0000100	-1.00000000	-.000010000	*	*
.0001000	-0.99999999	-.000100000	-.000000011	*
.0005000	-0.99999978	-.000500000	-.000000281	*
.0008000	-0.99999944	-.000799999	-.000000720	*
.0010000	-0.99999913	-.000999999	-.000001125	-.000000001
.0050000	-0.99997813	-.004999854	-.000028124	-.000000167
.0080000	-0.99994400	-.007999403	-.000071993	-.000000683
.0100000	-0.99991250	-.009998833	-.000112484	-.000001333
.0200000	-0.99965001	-.019990668	-.000449741	-.000010659
.0300000	-0.99921257	-.029968508	-.001011191	-.000035945

$s_1 = -1 + .875e^2$

$s_2 = -e + (7/6)e^3$

$s_3 = -(9/8)e^2$

$s_4 = (4/3)e^3$

Table B.13
Fourier Coefficient for F(4,4)

e	c_1	c_2	c_3	c_4
.0000001	-1.00000000	-.000000100	*	*
.0000010	-1.00000000	-.000001000	*	*
.0000100	-1.00000000	-.000010000	*	*
.0001000	-1.00000000	-.000100000	-.000000011	*
.0005000	-1.00000009	-.000500000	-.000000281	*
.0008000	-1.00000024	-.000800000	-.000000720	*
.0010000	-1.00000037	-.001000000	-.000001125	-.000000001
.0050000	-1.00000938	-.005000020	-.000028125	-.000000167
.0080000	-1.00002400	-.008000085	-.000072000	-.000000683
.0100000	-1.00003750	-.010000167	-.000112499	-.000001333
.0200000	-1.00015005	-.020001334	-.000449989	-.000010665
.0300000	-1.00033776	-.030004506	-.001012443	-.000035990

$c_1 = -1 - (3/8)e^2$

$c_2 = -e - (1/6)e^3$

$c_3 = -(9/8)e^2$

$c_4 = -(4/3)e^3$

Table B.14
Fourier Coefficient for F(5,5)

e	c_0	c_1	c_2	c_3	c_4
.0000001	-.000000150	1.00000010	.000000050	*	*
.0000010	-.000001500	1.00000100	.000000500	*	*
.0000100	-.000015000	1.00001000	.000005000	*	*
.0001000	-.000150015	1.00010001	.000050005	.000000004	*
.0005000	-.000750375	1.00050016	.000250125	.000000094	*
.0008000	-.001200961	1.00080040	.000400320	.000000240	*
.0010000	-.001501502	1.00100063	.000500500	.000000375	*
.0050000	-.007537688	1.00501570	.002512521	.000009422	.000000042
.0080000	-.012096774	1.00804032	.004032086	.000024192	.000000172
.0100000	-.151515152	1.01006332	.005050168	.000037875	.000000337
.0200000	-.030612245	1.02025511	.010201360	.000153004	.000002720
.0300000	-.046391753	1.03057992	.015454641	.000347645	.000009268

$c_0 = -1.5e - 1.5e^2 - 1.5e^3$

$c_1 = 1 + e + (5/8)e^2 + (5/8)e^3$

$c_2 = .5e + .5e^2 + .168e^3$

$c_3 = (3/8)e^2 + (3/8)e^3$

$c_4 = (1/3)e^3$

Table B.15
Fourier Coefficient for F(5,6)

e	s_1	s_2	s_3	s_4
.0000001	.999999900	.000000050	*	*
.0000010	.999999000	.000000500	*	*
.0000100	.999990000	.000005000	*	*
.0001000	.999899999	.000049995	.000000004	*
.0005000	.999499969	.000249875	.000000094	*
.0008000	.999199920	.000399680	.000000240	*
.0010000	.998999875	.000499500	.000000375	*
.0050000	.994996891	.002487479	.000009328	.000000041
.0080000	.991992064	.003967915	.000023807	.000000169
.0100000	.989987625	.004949835	.000037123	.000000330
.0200000	.979951001	.009798693	.000146967	.000002612
.0300000	.969890879	.014545635	.000327209	.000008724

$s_1 = 1 - e - (1/8)e^2 + (1/8)e^3$

$s_2 = .5e - .5e^2 - .165e^3$

$s_3 = (3/8)e^2 - (3/8)e^3$

$s_4 = (1/3)e^3$

Table B.16
Fourier Coefficient for F(6,5)

e	s_1	s_2	s_3	s_4
.0000001	-1.00000010	-.000000100	*	*
.0000010	-1.00000100	-.000001000	*	*
.0000100	-1.00001000	-.000010000	*	*
.0001000	-1.00010001	-.000100010	-.000000011	*
.0005000	-1.00050016	-.000500250	-.000000281	*
.0008000	-1.00080040	-.000800640	-.000000721	*
.0010000	-1.00100063	-.001001000	-.000001126	-.000000001
.0050000	-1.00501570	-.005025042	-.000028266	-.000000167
.0080000	-1.00804032	-.008064172	-.000072576	-.000000688
.0100000	-1.01006313	-.010100337	-.000113626	-.000001347
.0200000	-1.02025511	-.020402721	-.000459011	-.000010879
.0300000	-1.03057992	-.030909281	-.001042934	-.000037073

$s_1 = -1 - e - (5/8)e^2 - (5/8)e^3$

$s_2 = -e - e^2 - (1/3)e^3$

$s_3 = -(9/8)e^2 - (9/8)e^3$

$s_4 = -(4/3)e^3$

Table B.17
Fourier Coefficient for F(6,6)

e	c_1	c_2	c_3	c_4
.0000001	.999999900	.000000100	*	*
.0000010	.999999000	.000001000	*	*
.0000100	.999990000	.000010000	*	*
.0001000	.999899999	.000099990	.000000011	*
.0005000	.999499969	.000499750	.000000281	*
.0010000	.998999875	.000899000	.000001124	.000000001
.0050000	.994996890	.004974959	.000027984	.000000166
.0080000	.991992064	.007935831	.000071421	.000000677
.0100000	.989987625	.009899670	.000111369	.000001320
.0200000	.979951001	.019597387	.000440901	.000010450
.0300000	.969890879	.029091271	.000981628	.000034895

$$c_1 = 1 - e - (1/8)e^2 + (1/8)e^3$$
$$c_2 = e - e^2 - (1/3)e^3$$
$$c_3 = (9/8)e^2 + (9/8)e^3$$
$$c_4 = (4/3)e^3$$

Table B.18
Fourier Coefficient for y(1)

e	c_0	c_1	c_2	c_3	c_4
.0000001	-.999999933	1.33333307	.000000200	*	*
.0000010	-.999999333	1.33333067	.000002000	*	*
.0000100	-.999993333	1.33330667	.000020000	*	*
.0001000	-.999933337	1.33306668	.000199963	.000000028	*
.0005000	-.999666750	1.33200029	.000999083	.000000708	*
.0008000	-.999466880	1.33120075	.001597654	.000001811	.000000002
.0010000	-.999333666	1.33066783	.001996334	.000002828	.000000004
.0050000	-.996674938	1.32002913	.009908445	.000070209	.000000489
.0080000	-.994687746	1.31207450	.015765791	.000178775	.000001991
.0100000	-.993366171	1.30678301	.019634228	.000278337	.000003876
.0200000	-.986796065	1.28046414	.038540536	.001093392	.000030465
.0300000	-.980286828	1.25437504	.056724469	.002415313	.000100973

$c_0 = -1 + (2/3)e - (1/3)e^2 + .5e^3$

$c_1 = (4/3) - (8/3)e + (7/6)e^2 + (1/3)e^3$

$c_2 = 2e - (11/3)e^2 + (7/8)e^3$

$c_3 = 2.83e^2 - 4.5e^3$

$c_4 = 4e^3$

Table B.19
Fourier Coefficient for y(3)

e	c_0	c_1	c_2	c_3	c_4
.0000001	-.000000100	-2.00000000	-.000000300	*	*
.0000010	-.000001000	-2.00000000	-.000003000	*	*
.0000100	-.000010000	-2.00000000	-.000030000	*	*
.0001000	-.000100000	-2.00000002	-.000300000	-.000000042	*
.0005000	-.000500000	-2.00000044	-.001500000	-.000001062	*
.0008000	-.000800000	-2.00000112	-.002400001	-.000002720	-.000000003
.0010000	-.001000001	-2.00000175	-.003000001	-.000005250	-.000000006
.0050000	-.005000094	-2.00004375	-.015000167	-.000106250	-.000000740
.0080000	-.008000384	-2.00011201	-.000272002	-.000003029	-.000003029
.0100000	-.010000750	-2.00017501	-.030001333	-.000425005	-.000005917
.0200000	-.020006002	-2.00070022	-.060010670	-.001700078	-.000047330
.0300000	-.030020265	-2.00157609	-.090036030	-.003825393	-.000159723

$c_0 = -e - (3/4)e^2$

$c_1 = -2 - (7/4)e^2$

$c_2 = -3e - (4/3)e^3$

$c_3 = -(17/4)e^2$

$c_4 = -5.92e^3$

Table B.20
Fourier Coefficient for y(4)

e	s_1	s_2	s_3	s_4
.0000001	2.00000000	.000000300	*	*
.0000010	2.00000000	.000003000	*	*
.0000100	2.00000000	.000030000	*	*
.0001000	1.99999997	.000300000	.000000042	*
.0005000	1.99999931	.001499999	.000001062	*
.0008000	1.99999824	.002399998	.000002720	.000000003
.0010000	1.99999725	.002999995	.000004250	.000000006
.0050000	1.99993125	.014999396	.000106245	.000000740
.0080000	1.99982400	.023997525	.000271968	.000003029
.0100000	1.99972501	.029995167	.000424921	.000005915
.0200000	1.99890012	.059961340	.001698743	.000047294
.0300000	1.99752558	.089869549	.003818637	.000159449

$s_1 = 2 - (11/4)e^2$

$s_2 = 3e - 4.825e^3$

$s_3 = (17/4)e^2$

$s_4 = 5.92e^3$

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